§4.1 1ST Order Systems

previously, solved DE containing one dependent variable

since many common applications require >1 dependent variable, this will result in a system of simultaneous DE

note: independent variable will typically be time, t

prime notation will be assumed to be WRT t unless otherwise noted

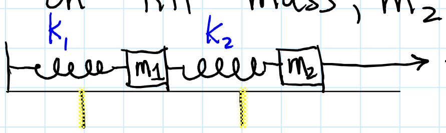
given: $f(t, x, y, x', y') = 0$
$g(t, x, y, x', y') = 0$

system of 2 1ST order DE w dep. variables x, y

a sol'n of this system is a pair: $x(t), y(t)$ that satisfy both eq'ns identically over some interval of t-values

SPRINGS

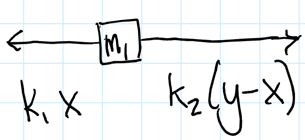
consider 2 masses, 2 springs, external force, $f(t)$, acting on RH mass, m_2



displacement of m_1 to the right is $x(t)$

displacement of m_2 is $y(t)$

static equilibrium i.e., springs are neither stretched nor compressed when x, y are zero



first spring stretched by x units

second spring stretched by $y-x$ units

then system would be:

$F = ma$

$$m_1 x'' = -k_1 x + k_2(y-x)$$
$$m_2 y'' = -k_2(y-x) + f(t)$$

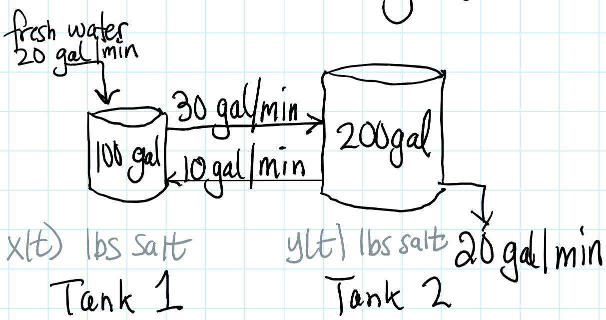
ex. given $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $f(t) = 40\sin 3t$

$$2x'' = -4x + 2(y-x)$$
$$\boxed{2x'' = -6x + 2y}$$
$$y'' = -2(y-x) + 40\sin 3t$$
$$\boxed{y'' = -2y + 2x + 40\sin 3t}$$

BRINE TANKS

Tank 1 contains $x(t)$ lbs of salt in 100 gal. of brine

Tank 2 " $y(t)$ " 200 gal "



Brine in each tank is stirred to maintain uniformity

brine is pumped between tanks

T1 $\longrightarrow$ T2 30 gal/min

T2 $\longrightarrow$ T1 10 gal/min

volume remains constant
- fresh water $\longrightarrow$ T1 20 gal/min
- brine exits T2 20 gal/min

define salt concentrations: Tank 1 $\frac{x}{100}$ $\frac{\text{lbs}}{\text{gal}}$

Tank 2 $\frac{y}{200}$ $\frac{\text{lbs}}{\text{gal}}$

compute salt's rate of change:

$$x' = -30\frac{x}{100} + 10\frac{y}{200} \qquad y' = 30 \cdot \frac{x}{100} - \underbrace{10 \cdot \frac{y}{200} - 20\frac{y}{200}}$$

$$20\left(x' = -\frac{3}{10}x + \frac{1}{20}y\right) \qquad 20\left(y' = \frac{3}{10}x - \frac{3}{20}y\right)$$

$$\boxed{\begin{aligned} 20x' &= -6x + y \\ 20y' &= 6x - 3y \end{aligned}}$$

strive to transform any higher order system into an equivalent system of 1st order eqns strive to transform any higher order system into an equivalent system of 1st order eqns

ex. 2nd order system format

$$x_1'' = f_1(t, x_1, x_2, x_1', x_2')$$
$$x_2'' = f_2(t, x_1, x_2, x_1', x_2')$$

generalize to nth order:

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

create let statements: $\boxed{x_1 = x} \xrightarrow{\text{deriv.}} (x_1)' = x' = x_2$

$\boxed{x_2 = x'} \rightarrow (x_2)' = x'' = x_3$

$x_3 = x''$

$\vdots$

$x_n = x^{(n-1)}$

etc,

rewrite: $(x_1)' = x_2$

$(x_2)' = x_3$

$\vdots$

$(x_{n-1})' = x_n$

then $(x_n)' = f(t, \underbrace{x_1, x_2, \dots, x_n}_{n \text{ 1st order eqns}})$

because $x^{n} = (x_n)'$

ex. given 3rd order eqn: $x''' + 3x'' + 2x' - 5x = \sin 2t$

solve for x''' $\quad x''' = 5x - 2x' - 3x'' + \sin 2t$

$$f(t, x, x', x'')$$

$x_1 = x$ $\quad x_2 = x' = (x_1)'$ $\quad x_3 = x'' = (x_2)'$

write as a system of 1st order eq'ns:

$(x_1)' = x_2$

write as a system of 1^{st} order eq'ns:

$(x_1)' = x_2$

$(x_2)' = x_3$

$x''' \rightarrow (x_3)' = 5x_1 - 2x_2 - 3x_3 + \sin 2t$

note: could've used Ch 3 techniques to solve eqn above, however, they would NOT have been applicable to a non-linear eqn such as: $x'' = x^3 + (x')^3$

using new technique:

$x = x_1$

$(x_1)' = x_2 \Rightarrow x'$

$(x_2)' = x_3 \Rightarrow x'' = (x_1)^3 + (x_2)^3$

revisit: $2x'' = -6x + 2y$
$y'' = 2x - 2y + 40\sin 3t$

$x_1 = x$
$x_2 = x' \Rightarrow (x_1)'$

$y_1 = y$
$y_2 = y' \Rightarrow (y_1)'$

create a system to define $(x_1)', (x_2)', (y_1)', (y_2)'$

$(x_2)' = x'' \Rightarrow$
$2(x_2)'' = -6x_1 + 2y_1$
$(x_1)' = x_2$
$(y_1)' = y_2$
$y'' \longrightarrow (y_2)' = 2x_1 - 2y_1 + 40\sin 3t$

have created 4 1^{st} order eq'ns using dependent variables x_1, x_2, y_1, y_2

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ex. linear 2^{nd} order DE: $x'' + px' + qx = 0$

assuming constant coefficients and indep. variable t would transform into 2-dimensional linear system

let $x' = y$
then $x'' = y'$

solve for x''

$\xrightarrow{\text{system}}$ $x' = y$

then $x'' = y'$ $\xrightarrow{\text{system}}$ $x' = y$

$x'' = -qx - px'$ $\quad y' = -qx - py$

y' $\quad y$

ex. solve 2-dimensional system: $x' = -2y$

$y' = \frac{1}{2}x$

$(x' = -2y)'$

$x'' = -2y'$

$x'' = -2\left(\frac{1}{2}x\right)$

$x'' = -x \quad \Rightarrow \quad x'' + x = 0$

char eqn: $r^2 + 1 = 0$

$r = \pm i$

$x(t) = A\cos t + B\sin t$

will complete this problem next time…